J. Math. Anal. Appl. 384 (2011) 677–682

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Subdifferential calculus for a quasiconvex function with generator

Satoshi Suzuki *, Daishi Kuroiwa

Interdisciplinary Faculty of Science and Engineering, Shimane University, Japan

ARTICLE INFO

Article history:
Received 14 March 2011
Available online 12 June 2011
Submitted by A. Dontchev

Keywords:
Quasiconvex function
Subdifferential
Monotonicity
Mean-value theorem

ABSTRACT

Recently, we discussed optimality conditions for quasiconvex programming by introducing 'Q-subdifferential', which is a notion of differential of quasiconvex functions. In this paper, we investigate basic and fundamental properties of the Q-subdifferential. Especially, we show results of a chain rule for composition with non-decreasing functions, monotonicity of the Q-subdifferential, mean-value theorem, a sufficient condition for a global minimizer for quasiconvex programming, and the calculus of the Q-subdifferential of the supremum of quasiconvex functions.

1. Introduction

In convex programming, the subdifferential, which is a generalized notion of the differential, plays very important roles to discuss optimality conditions. For example, it is well known that x_0 is a global minimizer of a convex function f in a closed convex set A if and only if $0 \in \partial f(x_0) + N_A(x_0)$, and this result is used extensively in various studies.

In quasiconvex programming, several types of subdifferentials have been defined and observed by many researchers, for example GP-subdifferential [3], R-quasi-subdifferential [14], MLS-subdifferential [7] and so on. In these literatures, properties of such subdifferentials of quasiconvex functions and some optimality conditions of quasiconvex programming which are similar to the above optimality condition of convex programming have been studied. However, these subdifferentials are not generalizations of the differential, that is, even if a quasiconvex function is differentiable, these subdifferentials are not equal to the differential.

Recently, we introduced the subdifferential for quasiconvex functions (the Q-subdifferential) by using the notion of generator in [13]. The Q-subdifferential is a generalization of the Gâteaux derivative and the subdifferential in the sense of convex analysis. Also, we discussed a necessary condition for a local minimizer and a constraint qualification for quasiconvex programming by using the Q-subdifferential.

In this paper, we investigate basic and fundamental properties of the Q-subdifferential. The remainder of the present paper is organized as follows. In Section 2, we introduce some preliminaries. From Section 3 to Section 7, we investigate a chain rule for composition with non-decreasing functions, monotonicity of the Q-subdifferential, a mean-value theorem with respect to the Q-subdifferential, a sufficient condition for a global minimizer by using the Q-subdifferential, and the Q-subdifferential of the supremum of quasiconvex functions.

2. Preliminaries

Let X be a locally convex Hausdorff topological vector space, let X^* be the continuous dual space of X, and let $\langle x^*, x\rangle$ denote the value of a functional $x^* \in X^*$ at $x \in X$. Given a set $A^* \subset X^*$, we denote the weak*-closure, the convex hull, the

* Corresponding author.
E-mail address: suzuki@math.shimane-u.ac.jp (S. Suzuki).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.06.015

conical hull, and the relative interior generated by A^*, by $\operatorname{cl} A^*$, $\operatorname{co} A^*$, $\operatorname{cone} A^*$, and $\operatorname{ri} A^*$, respectively. The indicator function δ_A of A is defined by

$$\delta_A(x) := \begin{cases} 0, & x \in A, \\ \infty, & \text{otherwise.} \end{cases}$$

Throughout the present paper, let f be a function from X to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = [-\infty, \infty]$. Here, f is said to be proper if for all $x \in X$, $f(x) > -\infty$ and there exists $x_0 \in X$ such that $f(x_0) \in \mathbb{R}$. We denote the domain of f by $\operatorname{dom} f$, that is, $\operatorname{dom} f = \{x \in X \mid f(x) < \infty\}$. The epigraph of f, $\operatorname{epi} f$, is defined as $\operatorname{epi} f = \{(x, r) \in X \times \mathbb{R} \mid f(x) \leqslant r\}$, and f is said to be convex if $\operatorname{epi} f$ is convex. In addition, the Fenchel conjugate of f, $f^* : X^* \to \overline{\mathbb{R}}$, is defined as $f^*(u) = \sup_{x \in \operatorname{dom} f}\{\langle u, x\rangle - f(x)\}$. Remember that f is said to be quasiconvex if for all $x_1, x_2 \in X$ and $\lambda \in (0, 1)$,

$$f\big((1-\lambda)x_1 + \lambda x_2\big) \leqslant \max\big\{f(x_1), f(x_2)\big\}.$$

Define the level sets of f with respect to a binary relation $\diamond$ on $\overline{\mathbb{R}}$ as

$$L(f, \diamond, \beta) = \big\{x \in X \bigm| f(x) \diamond \beta\big\}$$

for any $\beta \in \mathbb{R}$. Then, f is quasiconvex if and only if for any $\beta \in \mathbb{R}$, $L(f, \leqslant, \beta)$ is a convex set, or equivalently, for any $\beta \in \mathbb{R}$, $L(f, <, \beta)$ is a convex set. Any convex function is quasiconvex, but the opposite is not true.

It is well known that a proper lsc convex function consists of a supremum of some family of affine functions. In the case of quasiconvex functions, a similar result was also proved in [6,8]. First, we introduce a notion of quasiaffine function. A function f is said to be quasiaffine if quasiconvex and quasiconcave. It is worth noting that f is lsc quasiaffine if and only if there exist $k \in Q$ and $w \in X^*$ such that $f = k \circ w$, where $Q = \{h : \mathbb{R} \to \overline{\mathbb{R}} \mid h \text{ is lsc and non-decreasing}\}$. By using the notion of quasiaffine, it was proved that f is lsc quasiconvex if and only if there exists $\{(k_i, w_i) \mid i \in I\} \subset Q \times X^*$ such that $f = \sup_{i \in I} k_i \circ w_i$. This result indicates that an lsc quasiconvex function f consists of a supremum of some family of lsc quasiaffine functions. In [12], we define a notion of generator for quasiconvex functions, that is, $\{(k_i, w_i) \mid i \in I\} \subset Q \times X^*$ is said to be a generator of f if $f = \sup_{i \in I} k_i \circ w_i$. Because of the result in [6,8], all lsc quasiconvex functions have at least one generator. Also, when f is a proper lsc convex function, $B_f = \{(k_v, v) \mid v \in \operatorname{dom} f^*,\ k_v(t) = t - f^*(v),\ \forall t \in \mathbb{R}\} \subset Q \times X^*$ is a generator of f. Actually, for all $x \in X$,

$$f(x) = f^{**}(x) = \sup\big\{\langle v, x\rangle - f^*(v) \bigm| v \in \operatorname{dom} f^*\big\} = \sup_{v \in \operatorname{dom} f^*} k_v\big(\langle v, x\rangle\big).$$

We call the generator B_f "the basic generator" of convex function f. The basic generator is very important with respect to the comparison of convex and quasiconvex programming.

Also, we denote the lower left-hand Dini derivative of $h \in Q$ at t by $D_-h(t)$, that is $D_-h(t) = \liminf_{\varepsilon \to 0-} \frac{h(t+\varepsilon) - h(t)}{\varepsilon}$. A function h is said to be lower left-hand Dini differentiable if $D_-h(t)$ is finite for all $t \in \mathbb{R}$.

In [13], we introduced the following subdifferential for quasiconvex functions.

Definition 1. (See [13].) Let f be an lsc quasiconvex function with a generator $G = \{(k_i, w_i) \mid i \in I\} \subset Q \times X^*$, and assume that k_i is lower left-hand Dini differentiable for all $i \in I$. Then, we define the subdifferential for quasiconvex functions (the Q-subdifferential) of f at x_0 with respect to G as follows:

$$\partial_G f(x_0) = \operatorname{cl} \operatorname{co}\big\{D_-k_i\big(\langle w_i, x_0\rangle\big) w_i \bigm| i \in I(x_0)\big\},$$

where $I(x_0) = \{i \in I \mid f(x_0) = k_i \circ w_i(x_0)\}$.

The Q-subdifferential is a generalized notion of the subdifferential for convex functions. Actually, if f is a proper lsc convex function with the basic generator B_f, then

$$\begin{aligned} \partial_{B_f} f(x_0) &= \operatorname{cl} \operatorname{co}\big\{D_-k_v\big(\langle v, x_0\rangle\big) v \bigm| v \in \operatorname{dom} f^*,\ f(x_0) = k_v\big(\langle v, x\rangle\big)\big\} \\ &= \operatorname{cl} \operatorname{co}\big\{v \bigm| v \in \operatorname{dom} f^*,\ f(x_0) = \langle v, x\rangle - f^*(v)\big\} \\ &= \partial f(x_0). \end{aligned}$$

Also, if f is Gâteaux differentiable at x_0, k_s are differentiable at x_0 for all $i \in I(x_0)$, and $I(x_0) \neq \emptyset$, then we can check $\partial_G f(x_0) = \{f'(x_0)\}$, see [13].

Remark 1. In [3], Greenberg and Pierskalla introduced the quasi-subdifferential. In [7], Martínez-Legaz and Sach introduced the Q-subdifferential, and called the quasi-subdifferential Greenberg–Pierskalla subdifferential. In this paper, we call quasi-subdifferential in [3] GP-subdifferential, Q-subdifferential in [7] MLS-subdifferential, and the subdifferential in [13] the Q-subdifferential.

By using the Q-subdifferential, we investigated the following optimality condition for quasiconvex programming.

Theorem 1. *(See [13].) Let A be a closed convex subset of X, f be an lsc quasiconvex function with a generator $G = \{(k_i, w_i) \mid i \in I\} \subset Q \times X^*$ and $x_0 \in A$. Assume that k_i is lower left-hand Dini differentiable for all $i \in I$ and at least one of the following holds*:

(i) *I is finite and k_i is continuous for all $i \in I$,*
(ii) *X is a Banach space, I is a compact topological space, $i \mapsto w_i$ is continuous on I to $(X^*, \|\cdot\|)$, $(i,t) \mapsto k_i(t)$ is usc on $I \times \mathbb{R}$, and $(i,t) \mapsto D_-k_i(t)$ is continuous on $I \times \mathbb{R}$.*

If x_0 is a local minimizer of f in A then, $0 \in \partial_G f(x_0) + N_A(x_0)$.

3. Chain rule for composition with non-decreasing functions

The chain rule of the usual differential is well known, and the chain rule of the subdifferential in the sense of convex analysis is also investigated. In [7], Martínez-Legaz and Sach investigated a chain rule for the MLS-subdifferential. In this section, we investigate a chain rule for composition with non-decreasing functions.

Theorem 2. *Let f be a real-valued lsc quasiconvex function with a differentiable generator $G = \{(k_i, w_i) \mid i \in I\} \subset Q \times X^*$, $g \in Q$ be continuous and $x_0 \in X$. Assume that g is differentiable at $f(x_0)$, then*

$$g'\big(f(x_0)\big)\partial_G f(x_0) \subset \partial_{\bar{G}}(g \circ f)(x_0),$$

where $\bar{G} = \{(g \circ k_i, w_i) \mid i \in I\}$. Moreover, if g is increasing, then equality holds.

Proof. At first, we show that $\bar{G}$ is a generator of $g \circ f$. It is clear that $g \circ f = \sup_{i \in I} g \circ k_i \circ w_i$, and we can check $g \circ k_i$ is lsc. Actually, if $\{t_n\} \subset \mathbb{R}$ converges to $t \in \mathbb{R}$, then $k_i(t) \leqslant \liminf_{k \to \infty} k_i(t_n)$. Since g is non-decreasing and continuous,

$$g\big(k_i(t)\big) \leqslant \liminf_{k \to \infty} g\big(k_i(t_n)\big).$$

Hence, $\bar{G} \subset Q \times X^*$ is a generator of $g \circ f$. If $i \in I(x_0)$, then $g \circ f(x_0) = g \circ k_i \circ w_i(x_0)$, this implies that for all $v \in \{k_i'(\langle w_i, x_0\rangle)w_i \mid i \in I(x_0)\}$,

$$g'\big(f(x_0)\big)v \in \big\{(g \circ k_i)'\big(\langle w_i, x_0\rangle\big)w_i \;\big|\; g \circ f(x_0) = g \circ k_i \circ w_i(x_0)\big\},$$

that is, $g'(f(x_0))\partial_G f(x_0) \subset \partial_{\bar{G}}(g \circ f)(x_0)$. Moreover, if g is increasing, then "$i \in I(x_0)$" and "$g \circ f(x_0) = g \circ k_i \circ w_i(x_0)$" are equivalent. Hence, the equality holds. □

4. Monotonicity of the Q-subdifferential

In this section, we investigate monotonicity of the Q-subdifferential. It is well known that if a function f is proper lsc convex, then the subdifferential of f in the sense of convex analysis is maximal monotone. This result is fundamental and useful for convex programming problems, and has been studied extensively. Also, it is well known that a function f is convex (quasiconvex) if and only if the Clarke subdifferential of f is monotone (quasimonotone, respectively), in detail, see [5,10]. Some similar results were investigated by some researchers for the other subdifferentials.

In the following theorem, we show that the Q-subdifferential is quasimonotone, that is, for all $x, y \in X$, $x^* \in \partial_G f(x)$, $y^* \in \partial_G f(y)$,

$$\min\big\{\langle x^*, y - x\rangle, \langle y^*, x - y\rangle\big\} \leqslant 0.$$

Theorem 3. *Let f be a continuous quasiconvex function with a differentiable generator G. Then, $\partial_G f$ is quasimonotone.*

Proof. We assume that there exist $x, y \in X$, $x^* \in \partial_G f(x)$ and $y^* \in \partial_G f(y)$ such that $\langle x^*, y - x\rangle > 0$ and $\langle y^*, x - y\rangle > 0$. Then, there exist x_0^* and $y_0^* \in X^*$ such that $x_0^* \in \mathrm{co}\{k_i'(\langle w_i, x\rangle)w_i \mid i \in I(x)\}$, $y_0^* \in \mathrm{co}\{k_i'(\langle w_i, y\rangle)w_i \mid i \in I(y)\}$, and $\langle x_0^*, y - x\rangle > 0$ and $\langle y_0^*, x - y\rangle > 0$. Also, there exists $i_x \in I(x)$ such that $\langle w_{i_x}, y - x\rangle > 0$ and $k_{i_x}'(\langle w_{i_x}, x\rangle) > 0$. Since k_{i_x} is non-decreasing and differentiable, $k_{i_x}(\langle w_{i_x}, y\rangle) > k_{i_x}(\langle w_{i_x}, x\rangle)$. This implies $f(y) > f(x)$. Similarly, we can prove that $f(x) > f(y)$, this is a contradiction. □

5. Mean-value theorem

The mean-value theorem for usual differentiable functions is well known and studied extensively. In convex analysis, many researchers investigated some types of mean-value theorem, for example, see [1,2,4,9,11]. In this section, we investigate mean-value theorem with respect to the Q-subdifferential.

The following lemma is essential.

Lemma 1. *Let f be a continuous quasiconvex function from $\mathbb{R}$ to $\mathbb{R}$ with a differentiable generator $G = \{(k_i, w_i) \mid i \in I\} \subset Q \times \mathbb{R}$. Assume that at least one of the following holds*:

(i) *I is finite,*
(ii) *I is a compact topological space, $(i,t) \mapsto k_i(\langle w_i, t\rangle)$ is usc on $I \times \mathbb{R}$, and $(i,t) \mapsto k_i'(\langle w_i, t\rangle) w_i$ is continuous on $I \times \mathbb{R}$.*

Then, there exists $c \in (0,1)$ such that $f(1) - f(0) \in \partial_G f(c)$.

Proof. Let $\alpha = f(1) - f(0)$ and $F(t) = f(t) - \alpha t - f(0)$ for all $t \in [0,1]$, then, $F(0) = F(1) = 0$.

Case 1. There exists $c \in (0,1)$ such that $F(c) = \min_{t\in[0,1]} F(t)$.

Then, for all $x \in [0,1]$, $F(x) \geqslant 0(x-c) + F(c)$. Because of the definition of F, for all $x \in [0,1]$, $f(x) \geqslant \alpha(x-c) + f(c)$.

By the assumption, we can check

$$\partial_G f(c) = \Big[\min_{i\in I(c)} k_i'\big(\langle w_i, c\rangle\big) w_i, \max_{i\in I(c)} k_i'\big(\langle w_i, c\rangle\big) w_i\Big].$$

Indeed, if I is finite, then it is clear. If (ii) holds, we can check $I(c)$ is compact because $I(c) = \{i \in I \mid k_i(\langle w_i, c\rangle) = f(c)\} = \{i \in I \mid k_i(\langle w_i, c\rangle) \geqslant f(c)\}$. Because of the continuity of $k_i'(\langle w_i, c\rangle) w_i$ on I, the above equality holds.

For each $n \in \mathbb{N}$, $I(c + \frac{1}{n})$ is nonempty since I is compact and $k_i(\langle w_i, c\rangle)$ is usc on I (or I is finite). Without loss of generality, we can choose a sequence $\{i_n\} \in I$ such that $i_n \in I(c + \frac{1}{n})$ and i_n converges to some $i_0 \in I$, because of the compactness (or finiteness, respectively) of I. Then,

$$f(c) \leqslant \liminf_{n\to\infty} f\left(c + \frac{1}{n}\right) \leqslant \limsup_{n\to\infty} k_{i_n} \circ w_{i_n}\left(c + \frac{1}{n}\right) \leqslant k \circ w_{i_0}(c),$$

that is, $i_0 \in I(c)$. Also,

$$\begin{aligned}\frac{k_{i_n} \circ w_{i_n}(c + \frac{1}{n}) - k_{i_n} \circ w_{i_n}(c)}{c + \frac{1}{n} - c} &\geqslant \frac{f(c + \frac{1}{n}) - f(c)}{\frac{1}{n}} \\ &\geqslant \frac{\alpha(c + \frac{1}{n} - c)}{\frac{1}{n}} \\ &= \alpha.\end{aligned}$$

By using the usual mean-value theorem for $k_{i_n} \circ w_{i_n}$, there exists $c_n \in (c, c + \frac{1}{n})$ such that $(k_{i_n} \circ w_{i_n})'(c_n) \geqslant \alpha$. Since c_n converges to c,

$$(k_{i_0} \circ w_{i_0})'(c) = \lim_{n\to\infty} (k_{i_n} \circ w_{i_n})'(c_n) \geqslant \alpha.$$

This implies that $\max_{i\in I(c)} k_i'(\langle w_i, c\rangle) w_i \geqslant \alpha$. Similarly we can prove that $\alpha \geqslant \min_{i\in I(c)} k_i'(\langle w_i, c\rangle) w_i$, that is, $f(1) - f(0) = \alpha \in \partial_G f(c)$.

Case 2. For all $y \in (0,1)$, $F(y) > \min_{x\in[0,1]} F(x)$.

Then, $F(1) = F(0) = \min_{x\in[0,1]} F(x)$ and there exists $c \in (0,1)$ such that $F(c) = \max_{x\in[0,1]} F(x)$ since F is continuous. Because of the definition of F, for all $x \in [0,1]$, $f(x) \leqslant \langle f(1) - f(0), x - c\rangle + f(c)$. Then, for all $i \in I(c)$,

$$\begin{aligned}\liminf_{\varepsilon\to 0+} \frac{f(c) - f(c-\varepsilon)}{\varepsilon} &\leqslant \limsup_{\varepsilon\to 0+} \frac{f(c) - f(c-\varepsilon)}{\varepsilon} \\ &\leqslant \limsup_{\varepsilon\to 0+} \frac{k_i \circ w_i(c) - k_i \circ w_i(c-\varepsilon)}{\varepsilon} \\ &= (k_i \circ w_i)'(c) \\ &= \liminf_{\varepsilon\to 0+} \frac{k_i \circ w_i(c+\varepsilon) - k_i \circ w_i(c)}{\varepsilon} \\ &\leqslant \liminf_{\varepsilon\to 0+} \frac{f(c+\varepsilon) - f(c)}{\varepsilon}\end{aligned}$$

$$
\begin{aligned}
&\leqslant \limsup_{\varepsilon\to 0+} \frac{f(c+\varepsilon)-f(c)}{\varepsilon}\\
&\leqslant \limsup_{\varepsilon\to 0+} \frac{f(c)+\alpha(c+\varepsilon-c)-f(c)}{\varepsilon}\\
&=\alpha\\
&\leqslant \liminf_{\varepsilon\to 0+} \frac{f(c)-(f(c)+\alpha(c-\varepsilon-c))}{\varepsilon}\\
&\leqslant \liminf_{\varepsilon\to 0+} \frac{f(c)-f(c-\varepsilon)}{\varepsilon}.
\end{aligned}
$$

Hence, f is differentiable at c and $f'(c)=\alpha$. Because of the assumption, $I(c)\neq\emptyset$, that is,

$$f(1)-f(0)=\alpha\in\{f'(c)\}=\partial_G f(c). \qquad \square$$

Now we show the mean-value theorem for quasiconvex functions.

Theorem 4. *Let f be a continuous quasiconvex function with a differentiable generator $G=\{(k_i,w_i)\mid i\in I\}\subset Q\times X^*$, $x,y\in X$, and $x\neq y$. Assume that at least one of the following holds*:

(i) *I is finite,*
(ii) *X is a Banach space, I is a compact topological space, $(i,x)\mapsto k_i\circ w_i(x)$ is usc on $I\times X$, and $(i,x)\mapsto k_i'(\langle w_i,x\rangle)w_i$ is continuous on $I\times X$ to $(X^*,\sigma(X^*,X))$.*

Then, there exist $z\in(x,y)$ and $z^*\in\partial_G f(z)$ such that

$$f(y)-f(x)=\langle z^*,y-x\rangle.$$

Proof. Let g be the following function from $\mathbb{R}$ to $\mathbb{R}$, $g(t)=f((1-t)x+ty)$ for all $t\in\mathbb{R}$. Then, $\bar G=\{(\bar k_i,\langle w_i,y-x\rangle)\mid i\in I,\bar k_i(t)=k_i(t+\langle w_i,a\rangle)\}$ is a generator of g. We can check that $\partial_G f(z)$ is w^*-compact for each $z\in X$. Actually, if the assumption (i) holds, it is obvious. If the assumption (ii) holds, $\partial_G f(z)$ is w^*-closed and bounded. By using the Banach–Alaoglu theorem, $\partial_G f(z)$ is w^*-compact. Then, we can check that

$$\partial_{\bar G} g(t)=\{\langle z^*,y-x\rangle \mid z^*\in\partial_G f((1-t)x+ty)\}.$$

By the assumption, at least one of the conditions (i) or (ii) in Lemma 1 holds. Hence, by using Lemma 1, there exist $c\in(0,1)$ and $c^*\in\partial_{\bar G} f(c)$ such that $g(1)-g(0)=c^*$. Put $z=(1-c)x+cy$, then $c^*\in\{\langle z^*,y-x\rangle\mid z^*\in\partial_G f(z)\}$. Hence, there exists $z^*\in\partial_G f(z)$ such that $f(y)-f(x)=\langle z^*,y-x\rangle$. $\square$

Remark 2. Assumptions in Theorem 4 are similar to assumptions in Theorem 1. If f has a differentiable generator, the condition (ii) in Theorem 4 is weaker than the condition (ii) in Theorem 1. However, in Theorem 1, we assume that the generator is only lower left-hand Dini differentiable. Anyway, these assumptions are satisfied when f is an lsc convex function with the basic generator and $\operatorname{dom} f^*$ is compact. For this reason, it seems that these conditions are not so strong for quasiconvex programming.

6. Sufficient condition for a global minimizer

In convex analysis, equivalent conditions for a global solution were investigated by using the subdifferential. In quasiconvex analysis, many researchers investigated optimality conditions by using some subdifferentials. In [13], we investigate a necessary condition for a local solution by using the Q-subdifferential.

In this section, we show a sufficient condition for a global solution of quasiconvex programming problem.

Theorem 5. *Let f be an lsc quasiconvex function with a generator $G=\{(k_i,w_i)\mid i\in I\}\subset Q\times X^*$, $A\subset X$ be closed convex and $x_0\in A$. Assume that for all $i\in I(x_0)$, $D_-k_i(\langle w_i,x_0\rangle)>0$. If $0\in\operatorname{ri}\partial_G f(x_0)+N_A(x_0)$, then, $f(x_0)=\min_{x\in A} f(x)$.*

Proof. At first, we can see that $\operatorname{ri}\partial_G f(x_0)=\operatorname{ri}\operatorname{cl}\operatorname{co}\{D_-k_i(\langle w_i,x_0\rangle)w_i\mid i\in I(x_0)\}=\operatorname{ri}\operatorname{co}\{D_-k_i(\langle w_i,x_0\rangle)w_i\mid i\in I(x_0)\}\subset\operatorname{co}\{D_-k_i(\langle w_i,x_0\rangle)w_i\mid i\in I(x_0)\}$. If $0\in\operatorname{ri}\partial_G f(x_0)+N_A(x_0)$, there exists $x^*\in\operatorname{co}\{D_-k_i(\langle w_i,x_0\rangle)w_i\mid i\in I(x_0)\}$ such that $-x^*\in N_A(x_0)$. Hence, for all $x\in X$, there exists $i\in I(x_0)$ such that $D_-k_i(\langle w_i,x_0\rangle)\langle w_i,x-x_0\rangle\geqslant 0$. Since, k_i is non-decreasing and $D_-k_i(\langle w_i,x_0\rangle)>0$, $f(x)\geqslant k_i\circ w_i(x)\geqslant k_i\circ w_i(x_0)=f(x_0)$. This completes the proof. $\square$

Also, if $\mathrm{co}\{D_-k_i(\langle w_i, x_0\rangle)w_i \mid i \in I(x_0)\}$ is w^*-closed and $D_-k_i(\langle w_i, x_0\rangle) > 0$ for all $i \in I(x_0)$, then $0 \in \partial_G f(x_0) + N_A(x_0)$ implies $f(x_0) = \min_{x\in A} f(x)$. If f is a proper lsc convex function with basic generator, then these assumptions satisfy, and the following well-known equivalence relation holds:

$$0 \in \partial f(x_0) + N_A(x_0) \quad \Longleftrightarrow \quad f(x_0) = \min_{x\in A} f(x).$$

7. The Q-subdifferential of the supremum of quasiconvex functions

Theorem 6. *Let I be an index set, for each $i \in I$, g_i be an lsc quasiconvex function from X to $\overline{\mathbb{R}}$ with a generator $G_i = \{(k_j^i, w_j^i) \mid j \in J_i\} \subset Q \times X^*$, $g = \sup_{i\in I} g_i$, $G = \bigcup_{i\in I} G_i$, $x_0 \in X$ and $I(x_0) = \{i \in I \mid g(x_0) = g_i(x_0)\}$. Then, following conditions hold*:

(i) *G is a generator of g,*
(ii) $\partial_G g(x_0) = \mathrm{cl}\,\mathrm{co} \bigcup_{i\in I(x_0)} \partial_{G_i} g(x_0)$.

Proof. It is clear that (i) holds, and we only show the condition (ii). Let $v \in \{D_-k(\langle w, x_0\rangle)w \mid (k, w) \in G,\ g(x_0) = k \circ w(x_0)\}$, then, there exists $(k_0, w_0) \in G$ such that

$$v = D_-k_0\big(\langle w_0, x_0\rangle\big)w_0 \quad \text{and} \quad g(x_0) = k_0 \circ w_0(x_0).$$

Since $G = \bigcup_{i\in I} G_i$, there exists $i_0 \in I$ such that $(k, w) \in G_i$. Hence,

$$g(x_0) \geqslant g_{i_0}(x_0) \geqslant k_{i_0} w_{i_0}(x_0) = g(x_0),$$

that is, $i_0 \in I(x_0)$ and $v \in \partial_{G_{i_0}} g_i(x_0)$. This implies that

$$\partial_G g(x_0) \subset \mathrm{cl}\,\mathrm{co} \bigcup_{i\in I(x_0)} \partial_{G_i} g(x_0).$$

Conversely, for all $i \in I(x_0)$ and $v \in \{D_-k_j^i(\langle w_j^i, x_0\rangle)w_j^i \mid j \in J_i,\ \ g(x_0) = k_j^i \circ w_j^i(x_0)\}$, there exists $j_0 \in J_i$ such that $v = D_-k_{j_0}^i(\langle w_{j_0}^i, x_0\rangle)w_{j_0}^i$ and $g(x_0) = k_{j_0}^i \circ w_{j_0}^i(x_0)$. Since $i \in I(x_0)$, $g(x_0) = g_i(x_0) = k_{j_0}^i \circ w_{j_0}^i(x_0)$, this implies the converse inclusion. $\square$

Acknowledgment

The authors are grateful to anonymous referee for many comments and suggestions improved the quality of the paper.

References

[1] J.M. Borwein, A note on ε-subgradients and maximal monotonicity, Pacific J. Math. 103 (1982) 307–314.
[2] F.H. Clarke, Yu.S. Ledyaev, Mean value inequalities, Proc. Am. Math. Soc. 122 (1994) 1075–1083.
[3] H.J. Greenberg, W.P. Pierskalla, Quasi-conjugate functions and surrogate duality, Cah. Cent. Étud. Rech. Opér. 15 (1973) 437–448.
[4] V. Jeyakumar, D.T. Luc, Nonsmooth calculus, minimality, and monotonicity of convexificators, J. Optim. Theory Appl. 101 (1999) 599–621.
[5] D.T. Luc, Characterizations of quasiconvex functions, Bull. Aust. Math. Soc. 48 (1993) 393–406.
[6] J.E. Martínez-Legaz, Quasiconvex duality theory by generalized conjugation methods, Optimization 19 (1988) 603–652.
[7] J.E. Martínez-Legaz, P.H. Sach, A new subdifferential in quasiconvex analysis, J. Convex Anal. 6 (1999) 1–11.
[8] J.P. Penot, M. Volle, On quasi-convex duality, Math. Oper. Res. 15 (1990) 597–625.
[9] J.P. Penot, Mean-value theorem with small subdifferentials, J. Optim. Theory Appl. 94 (1997) 209–221.
[10] R.T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, New Jersey, 1970.
[11] M. Studniarski, Mean value theorems and sufficient optimality conditions for nonsmooth functions, J. Math. Anal. Appl. 111 (1985) 313–326.
[12] S. Suzuki, D. Kuroiwa, On set containment characterization and constraint qualification for quasiconvex programming, J. Optim. Theory Appl. 149 (2011) 554–563.
[13] S. Suzuki, D. Kuroiwa, Optimality conditions and the basic constraint qualification for quasiconvex programming, Nonlinear Anal. 74 (2011) 1279–1285.
[14] P.T. Thach, Diewert–Crouzeix conjugation for general quasiconvex duality and applications, J. Optim. Theory Appl. 86 (1995) 719–743.